ISRAEL JOURNAL OF MATHEMATICS **93** (1996), 93–112

LIMIT LAWS OF ENTRANCE TIMES FOR HOMEOMORPHISMS OF THE CIRCLE

BY

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ABSTRACT

Given a homeomorphism f of the circle with irrational rotation number and a descending chain of renormalization intervals J_n of f, we consider for each interval the point process obtained by marking the times for the orbit of a point in the circle to enter J_n. Assuming the point is randomly chosen by the unique invariant probability measure of f, we obtain necessary and sufficient conditions which guarantee convergence in law of the corresponding point process and we describe all the limiting processes. These conditions are given in terms of the convergent subsequences of the orbit of the rotation number of f under the Gauss transformation and under a certain realization of its natural extension. We also consider the case when the point is randomly chosen according to Lebesgue measure, f being a diffeomorphism which is C^1-conjugate to a rotation, and we show that the same necessary and sufficient conditions guarantee convergence in this case.

Introduction

Limit laws of entrance times have been obtained in various contexts such as: hyperbolic automorphisms of the torus and Markov chains [Pi], Axiom A diffeomorphisms and shifts of finite type with a Hölder potential [Hi], and piecewise expanding maps of the circle [CG] (see also [CC]). The general setting for the problem is as follows. Given an ergodic dynamical system $(X, \mathcal{B}, \mu, f)$ and a set

* This work is part of Projeto Temático de Equipe "Transição de Fase Dinâmica em Sistemas Evolutivos" financially supported by FAPESP grant 90/3918-5.
Received March 16, 1994

$A \in \mathcal{B}$ of positive measure, we consider the times $k > 0$ such that $f^k(x) \in A$, for each $x \in X$. These are called the **entrance** times* of the orbit of x to the set A. We define a point process τ_A on $(0, \infty)$ by assigning to each x the sum of point masses at these entrance times. The problem consists of finding conditions under which this process, after rescaling by some constant depending on A, converges in law, when $\mu(A)$ tends to zero. Since the expectation of the first entrance time is of the order $1/\mu(A)$, it is natural to rescale the process by this factor.

In the case of hyperbolic automorphisms of the torus with Haar measure and Axiom A diffeomorphisms with the Bowen–Ruelle measure, the limit of the rescaled entrance time process is proven to be a Poisson point process of constant rate 1, where A is taken in the neighbourhood basis of a point in the manifold, for almost every base point (cf. [Pi] for a preliminary result in this direction, and [Hi] for the general case). In the case of piecewise expanding maps of the circle with an absolutely continuous invariant measure, a Poisson limit law (with rate 1) is also obtained when A is taken in a sequence of intervals with diverging time of self-intersection (cf. [CG]). In all of these cases the property of exponential decay of correlations, which is an exponential mixing property, is crucial in order to prove this universal behaviour.

The purpose of this paper is to show that there is no universal limit law in the case of a homeomorphism of the circle with irrational rotation number α and with its unique invariant probability measure. Taking the set A in the sequence of renormalization intervals J_n of the homeomorphism, we obtain a necessary and sufficient condition on a subsequence of the J_n's which guarantees convergence of the corresponding rescaled entrance time point process. These subsequences are shown to be directly related to the convergent subsequences of the orbit of α under the Gauss transformation G and also under a certain realization of the natural extension of G. Using this correspondence we deduce that, for Lebesgue almost every rotation number α, there are an uncountable number of subsequences of the renormalization intervals of the homeomorphism which give pairwise different limit laws. The possible limits are either the stationary modified renewal process with first renewal distribution given by the uniform distribution on the unit interval (known as the lattice process); or a *non-stationary* process with dis-

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* Entrance times are commonly used to denote the times $k > 0$ for which $f^k(x) \in A$ and $f^{k-1}(x) \notin A$. However, in our case these times will coincide with the present definition, when $\mu(A)$ is sufficiently small.

tribution of intermediate renewal times given in Figures 1 and 2, for some value of the parameters $\theta \in (0,1]$ and $\omega \in [0,1)$.

We also consider the case of a diffeomorphism which is C^1-conjugate to an irrational rotation but with Lebesgue measure, and we show that the same results hold (up to scale change) for the convergence of the corresponding entrance time point process, the latter being rescaled by the inverse of the length of J_n.

1. Preliminaries and statements of results

Let G: $[0,1] \to [0,1]$ be the Gauss transformation given by $G(\alpha) = 1/\alpha - a_0(\alpha)$ for $\alpha > 0$, and $G(0) = 0$, where $a_0(\alpha) = [1/\alpha]$ is the greatest integer $\leq 1/\alpha$. We also set $a_0(0) = \infty$ and make the conventions that $1/0 = \infty$ and $1/\infty = 0$. Defining $a_n = a_n(\alpha) = a_0(G^n(\alpha))$ for all $n \geq 0$, we obtain the continued fraction expansion of α,

$$\alpha = [a_0, a_1, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots}}} .$$

We write $p_0 = p_0(\alpha) = 0$, $q_0 = q_0(\alpha) = 1$ and for $n \geq 1$ we let $p_n/q_n = p_n(\alpha)/q_n(\alpha) = [a_0, a_1, \ldots, a_{n-1}]$ denote the truncated expansion of α of order n in its irreducible form. It is well-known that

$$\begin{cases} q_{n+1} = a_n q_n + q_{n-1} , \\ p_{n+1} = a_n p_n + p_{n-1} , \end{cases}$$

for $n \geq 1$. The distance of a real number x to the nearest integer will be denoted by $\|x\|$. We shall also use the so-called double Gauss transformation Γ: $[0,1]^2 \to [0,1]^2$, which is a realization of the natural extension of G (cf. [IN]), defined by

$$\Gamma(\alpha, \beta) = \left(G(\alpha), \frac{1}{a_0(\alpha) + \beta} \right) .$$

Note that for $n \geq 1$,

$$\Gamma^n(\alpha, \beta) = \left(G^n(\alpha), [a_{n-1}, a_{n-2}, \ldots, a_0, b_0, b_1, \ldots] \right) ,$$

where $b_j = a_0(G^j(\beta))$ for $j \geq 0$, and hence the convergent subsequences of $\Gamma^n(\alpha, \beta)$ for $n \geq 0$ do not depend on β.

Now let f: $S^1 \to S^1$ be an orientation preserving homeomorphism of the circle without periodic points and let $\alpha = \alpha(f) \in [0,1)$ be its irrational rotation number. In what follows we shall omit the dependence on α and write simply $a_n(\alpha) = a_n$, $p_n(\alpha) = p_n$ and $q_n(\alpha) = q_n$.

We fix $z \in S^1$ and define $J_n \subseteq S^1$ as the closed interval of endpoints $f^{q_n}(z)$ and $f^{q_{n-1}}(z)$ containing z in its interior. We also define $I_n \subseteq J_n$ to be the closed subinterval of endpoints z and $f^{q_n}(z)$. Recall, from the basic properties of homeomorphisms of the circle, that $J_{n+1} \subseteq J_n$, and inside J_1 the points $f^{q_n}(z)$ and $f^{q_{n-1}}(z)$ lie on opposite sides of z for all $n > 1$.

For any subset $A \subseteq S^1$, we consider a sequence of maps $N_A^{(k)}\colon S^1 \to \mathbb{N}$, $k = 0, 1, \ldots$, given inductively by $N_A^{(0)} \equiv 0$ and

$$N_A^{(k)}(x) = \min\left\{j > N_A^{(k-1)}(x)\colon\ f^j(x) \in A\right\},$$

for all $x \in S^1$ and all $k \geq 1$. We call $N_A^{(k)}(x)$ the k^{th} **entrance time** of x in A. Let μ be the unique ergodic invariant probability measure for f. For $A = J_n$ the entrance times $N_n^{(k)} = N_{J_n}^{(k)}$ define the distribution functions

$$F_n^{(k)}(t) = \mu\left\{x \in S^1\colon\ \mu(J_n)\left(N_n^{(k)}(x) - N_n^{(k-1)}(x)\right) \leq t\right\}.$$

We now state our results concerning the limiting behaviour of these distributions.

THEOREM I: *For each subsequence $\sigma = \{n_i\}$ of $\mathbb{N}$, the corresponding distribution functions $F_{n_i}^{(1)}$ converge (pointwise or uniformly) if and only if either*

(a) *$G^{n_i}(\alpha) \to 0$, in which case the limit distribution is the uniform distribution on the unit interval; or*

(b) *$\Gamma^{n_i}(\alpha, \cdot) \to (\theta, \omega)$ for some $\theta > 0$ and $\omega < 1$, in which case the limit distribution is the continuous piecewise linear function $F_\sigma^{(1)}$ given by Figure 1.*

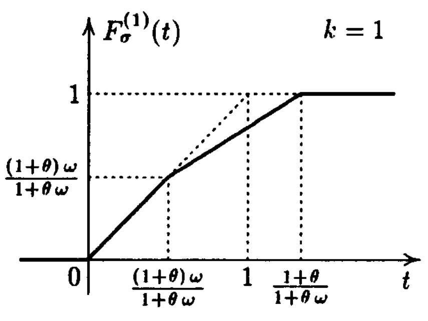

Figure 1

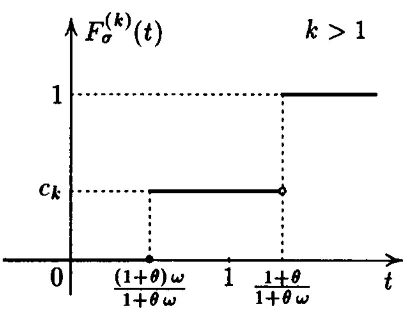

Figure 2

THEOREM II: *Let $\sigma = \{n_i\}$ be a subsequence of $\mathbb{N}$ such that $F^{(1)}_{n_i}$ converges (pointwise or uniformly). Then for each $k > 1$ the corresponding sequence of distribution functions $F^{(k)}_{n_i}$ converges uniformly. Furthermore*

(a) *If $G^{n_i}(\alpha) \to 0$ then the limit distribution is the distribution of the constant random variable $X \equiv 1$ for all $k > 1$;*

(b) *If $\Gamma^{n_i}(\alpha, \cdot) \to (\theta, \omega)$ for some $\theta > 0$ and $\omega < 1$, then the limit distribution is the step function $F^{(k)}_{\sigma}$ given by Figure 2, where*

$$c_k = \frac{\theta\,\omega}{1+\theta\,\omega} + \frac{1-\omega}{1+\theta\,\omega} \min\left\{\theta\,,\,(1+\theta)\left\|\frac{k\,\theta}{1+\theta}\right\|\right\}.$$

Remark A: Recall that G preserves an ergodic absolutely continuous invariant measure on the unit interval. Therefore, for Lebesgue almost every α, condition (a) holds in both Theorems for **some** subsequence σ. In other words, the uniform distribution occurs as a limit distribution along a subsequence for almost every α.

Remark B: Less well-known is the fact that Γ preserves an ergodic absolutely continuous invariant measure with respect to Lebesgue measure on the unit square, whose density is given by $1/\big((1+\alpha\,\beta)^2 \log 2\big)$ (cf. [IN, MP]). Thus, for Lebesgue almost every (α, β) and every fixed (θ, ω), there exists a subsequence $\sigma = \{n_i\}$ such that $\lim \Gamma^{n_i}(\alpha, \beta) = (\theta, \omega)$. However, from the properties of Γ stated at the beginning, we know that $\lim \Gamma^{n_i}(\alpha, \beta) = \lim \Gamma^{n_i}(\alpha, \alpha)$. This shows that for Lebesgue almost every α, condition (b) holds in both Theorems for **some** subsequence σ. In other words, unusual distributions such as the ones in Figures 1 and 2 occur as limit distributions along a subsequence for almost every α.

Now we interpret our results in terms of point processes. Let $\mathcal{M}[0, \infty)$ denote the σ-finite measures on $[0, \infty)$ and denote by δ_t the Dirac measure at the point t. For each $n \geq 1$, we define the point process $\tau_n \colon S^1 \to \mathcal{M}[0, \infty)$ by

$$\tau_n(x) = \sum_{k \geq 1} \delta_{\mu(J_n)\, N^{(k)}_n(x)}\,,$$

where x is assumed to be randomly chosen by μ. This is the point process of successive entrances to J_n (rescaled by $\mu(J_n)$).

THEOREM III: *For each subsequence $\sigma = \{n_i\}$ of $\mathbb{N}$, the point process of successive entrances to J_{n_i} converges if and only if either*

(a) *$G^{n_i}(\alpha) \to 0$, in which case the limit is the stationary modified renewal process known as the lattice process; or*

(b) $\Gamma^{n_i}(\alpha,\cdot) \to (\theta,\omega)$ *for some* $\theta > 0$ *and* $\omega < 1$, *in which case the limit is a* non-stationary *point process injectively parametrised by* θ *and* ω.

Next we let f be a diffeomorphism of the circle which is C^1-conjugate to a rotation with irrational rotation number α. In this case the unique invariant probability measure μ for f is absolutely continuous and has a continuous density $g(x)$. We also recall that $g(x) > 0$ for all $x \in S^1$. Consider the intervals J_n and the entrance time functions $N_n^{(k)}$ as before, and define the distributions with respect to Lebesgue measure by

$$\Phi_n^{(k)}(t) = \lambda\left\{x \in S^1\colon\ |J_n|\left(N_n^{(k)}(x) - N_n^{(k-1)}(x)\right) \le t\right\}$$

for each $k > 0$, where λ denotes Lebesgue measure on S^1 and $|J_n|$ means the length of J_n. In this context we have the following results.

THEOREM IV: *Let* f *be* C^1*-conjugate to the rotation by* α *and let* $\sigma = \{n_i\}$ *be a subsequence of* $\mathbb{N}$. *The distribution* $\Phi_{n_i}^{(1)}$ *converges (pointwise or uniformly) if and only if either*

(a) $G^{n_i}(\alpha) \to 0$, *in which case the limit distribution is the uniform distribution on the interval* $[0, 1/g(z)]$; *or*

(b) $\Gamma^{n_i}(\alpha,\cdot) \to (\theta,\omega)$ *for some* $\theta > 0$ *and* $\omega < 1$, *in which case the limit distribution is* $\Phi_\sigma^{(1)}(t) = F_\sigma^{(1)}\big(g(z)\,t\big)$ *where* $F_\sigma^{(1)}$ *is given by Figure 1.*

THEOREM V: *Let* $\sigma = \{n_i\}$ *be a subsequence of* $\mathbb{N}$ *such that* $\Phi_{n_i}^{(1)}$ *converges (pointwise or uniformly). Then for each* $k > 1$ *the corresponding sequence of distribution functions* $\Phi_{n_i}^{(k)}$ *converges uniformly. Moreover*

(a) *If* $G^{n_i}(\alpha) \to 0$ *then the limit distribution is the distribution of the constant random variable* $X \equiv 1/g(z)$ *for all* $k > 1$;

(b) *If* $\Gamma^{n_i}(\alpha,\cdot) \to (\theta,\omega)$ *for some* $\theta > 0$ *and* $\omega < 1$, *then the limit distribution is the step function* $\Phi_\sigma^{(k)}(t) = F_\sigma^{(k)}\big(g(z)\,t\big)$ *where* $F_\sigma^{(k)}$ *is given by Figure 2.*

Remark C: In terms of point processes, we can define the point process τ_n^λ of successive entrances to J_n (rescaled by $|J_n|$), where the randomness is given by a point in the circle chosen according to Lebesgue measure. A similar statement to Theorem III can be made for the convergence of τ_n^λ, where the conditions for convergence along a subsequence are the same in this case, and the point processes which appear as limits are rescaled by $1/g(z)$.

2. The entrance time functions

Let $n \geq 1$ be fixed. Consider the first return map $T\colon J_n \to J_n$ given by

$$T(x) = f^{N_n^{(1)}(x)}(x) .$$

One can show that $T(x) = f^{q_{n-1}}(x)$ if $x \in I_n$ and $T(x) = f^{q_n}(x)$ if $x \in I_{n-1} \setminus \{z\}$ (cf. [La]). Now we define the itinerary of a point $x \in J_n$ to be the sequence $\{\varepsilon_1(x), \varepsilon_2(x), \ldots\}$ where

$$\varepsilon_j(x) = \begin{cases} 0 & \text{if } T^j(x) \in I_n ; \\ 1 & \text{if } T^j(x) \in I_{n-1} . \end{cases}$$

It follows that for all $k \geq 1$ and $x \in J_n$ we have

$$N_n^{(k)}(x) = \sum_{j=1}^{k} \big((1 - \varepsilon_j(x))q_{n-1} + \varepsilon_j(x)q_n\big) . \tag{1}$$

LEMMA 1: *The intervals $\{f^i(I_n)\}$ for $i = 0, \ldots, q_{n-1} - 1$ together with the intervals $\{f^j(I_{n-1})\}$ for $j = 0, \ldots, q_n - 1$ cover the circle. If $I \neq J$ are any two of these intervals then either $I \cap J$ is empty or consists of a single point. In particular, $\mu(I \cap J) = 0$.*

Proof: First suppose f is the rotation by α. Let $0 \leq i < q_n$ and $0 \leq j < q_{n-1}$ satisfy $f^i(I_{n-1}) \cap f^j(I_n) \neq \emptyset$. If $i < j$ then $j - i < q_{n-1}$ and there exists $x \in I_n$ such that $f^{j-i}(x) \in I_{n-1}$, which contradicts the expression of T. If $j < i$ then $i - j < q_n$ and so the intersection $f^{i-j}(I_{n-1}) \cap I_n$ is necessarily z. This shows that the intervals of the first collection have disjoint interiors from the ones in the second collection. A similar argument proves that any two intervals from the same collection also have this property.

Since in this case μ is Lebesgue measure, these intervals cover the circle because their total measure is

$$q_n \left|I_{n-1}\right| + q_{n-1} \left|I_n\right| = q_n \left|q_{n-1}\alpha - p_{n-1}\right| + q_{n-1} \left|q_n\alpha - p_n\right| ,$$

which is equal to one due to a simple computation using the Lagrange equality $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$.

In the general case, let R_α denote the rotation by α and let h be the semi-conjugacy between f and R_α. Then the first assertion of the Lemma is true, since

h is onto. On the other hand, the pre-image of a point under h is either a point or a closed interval whose interior is a wandering component of f. In both cases, the μ-measure of this pre-image is zero because μ is non-atomic and its support is the minimal set of f. Now since the endpoints of the intervals in both collections belong to the set $\{z, f(z), \ldots, f^{q_n+q_{n-1}-1}(z)\}$ and this set is mapped bijectively by h onto the set $\{h(z), R_\alpha(h(z)), \ldots, R_\alpha^{q_n+q_{n-1}-1}(h(z))\}$, we conclude that if I and J are as stated then $I \cap J$ must have empty interior. ∎

As a direct consequence of Lemma 1 we have the following.

PROPOSITION 2: *The first entrance time function is given by*

$$N_n^{(1)}(x) = \begin{cases} q_{n-1} - i & \text{if } x \in f^i(I_n), 0 \le i < q_{n-1} ; \\ q_n - j & \text{if } x \in f^j(I_{n-1}) \setminus \{f^j(z)\}, 0 \le j < q_n . \end{cases}$$

PROPOSITION 3: *The distribution function of the first entrance time is given by*

$$F_n^{(1)}(t) = \begin{cases} k\,\mu(J_n) & \text{if } k\,\mu(J_n) \le t < (k+1)\mu(J_n), \\ & \quad 0 \le k < q_{n-1}; \\ (k - q_{n-1})\mu(I_{n-1}) + q_{n-1}\,\mu(J_n) & \text{if } k\,\mu(J_n) \le t < (k+1)\mu(J_n), \\ & \quad q_{n-1} \le k < q_n; \end{cases}$$

and $F_n^{(1)}(t) = 0$ if $t < 0$ and $F_n^{(1)}(t) = 1$ if $t \ge q_n\,\mu(J_n)$.

However, describing the distribution functions of the subsequent entrance times requires a different argument.

PROPOSITION 4: *For $k > 1$ the distribution function of the k-th entrance time is given by*

$$F_n^{(k)}(t) = q_{n-1}\,\mu(I_n \cap T^{-k} I_n) + q_n\,\mu(I_{n-1} \cap T^{-k} I_n) \tag{2}$$

if $q_{n-1}\,\mu(J_n) \le t < q_n\,\mu(J_n)$; $F_n^{(k)}(t) = 0$ if $t < q_{n-1}\,\mu(J_n)$; and $F_n^{(k)}(t) = 1$ if $t \ge q_n\,\mu(J_n)$.

Proof: We note that the difference

$$D_n^{(k)}(x) = N_n^{(k)}(x) - N_n^{(k-1)}(x)$$

is either q_{n-1} or q_n for $k > 1$, since these are the return times on J_n. Therefore $F_n^{(k)}(t)$ is a step function and it is clear that $F_n^{(k)}(t) = 0$ if $t < q_{n-1}\,\mu(J_n)$ and

$F_n^{(k)}(t) = 1$ if $t \geq q_n\,\mu(J_n)$. Hence it remains to compute the measure of the set E of points $x \in S^1$ where $D_n^{(k)}(x) = q_{n-1}$. By (1) the latter equality holds for $x \in J_n$ if and only if $T^k(x) \in I_n$. Thus $E \cap J_n = T^{-k} I_n$.

Now let x be a point outside J_n. Then from Lemma 1 we know that either $x \in f^i(I_n)$ for some $i = 1, \dots, q_{n-1}-1$ or $x \in f^j(I_{n-1})$ for some $j = 1, \dots, q_n - 1$, except for a set of measure zero. If $x \in f^i(I_n)$ then by Proposition 2 we have

$$D_n^{(k)}(x) \;=\; D_n^{(k-1)}(f^{q_n - i}(x))\,.$$

Therefore in this case, using (1) we conclude that $D_n^{(k)}(x) = q_{n-1}$ if and only if $f^{q_{n-1}-i}(x) \in T^{-(k-1)} I_n$. Since $f^{q_{n-1}-i}(f^i(I_n)) = T(I_n)$ we obtain

$$\mu(E \cap f^i(I_n)) \;=\; \mu(T^{-(k-1)} I_n \cap T(I_n)) \;=\; \mu(T^{-k} I_n \cap I_n)\,,$$

where we have used the fact that both T and f preserve μ.

In the case $x \in f^j(I_{n-1})$ a similar argument shows that

$$\mu(E \cap f^j(I_{n-1})) \;=\; \mu(T^{-(k-1)} I_n \cap T(I_{n-1})) \;=\; \mu(T^{-k} I_n \cap I_{n-1})\,.$$

Putting these facts together we finally obtain

$$\begin{aligned} \mu(E) &= \mu(E \cap J_n) + \sum_{i=1}^{q_{n-1}-1} \mu(E \cap f^i(I_n)) + \sum_{j=1}^{q_n - 1} \mu(E \cap f^j(I_{n-1})) \\ &= \mu(T^{-k} I_n) + (q_{n-1} - 1)\,\mu(I_n \cap T^{-k} I_n) + (q_n - 1)\,\mu(I_{n-1} \cap T^{-k} I_n) \\ &= q_{n-1}\,\mu(I_n \cap T^{-k} I_n) + q_n\,\mu(I_{n-1} \cap T^{-k} I_n)\,. \qquad \blacksquare \end{aligned}$$

Similarly to Proposition 3, the first result below is a direct consequence of Lemma 1, whereas the second is a consequence of the proof of Proposition 4.

PROPOSITION 5: *The distribution function with respect to Lebesgue measure of the first entrance time is given by*

$$\Phi_n^{(1)}(t) \;=\; \begin{cases} \displaystyle\sum_{i=0}^{k} |f^i(J_n)| & \text{if } k\,|J_n| \leq t < (k+1)|J_n|, \\ & \quad 0 \leq k < q_{n-1}; \\ \displaystyle\sum_{j=q_{n-1}}^{k} |f^j(I_{n-1})| + \sum_{i=0}^{q_{n-1}-1} |f^i(J_n)| & \text{if } k\,|J_n| \leq t < (k+1)|J_n|, \\ & \quad q_{n-1} \leq k < q_n; \end{cases}$$

and $\Phi_n^{(1)}(t) = 0$ *if* $t < 0$ *and* $\Phi_n^{(1)}(t) = 1$ *if* $t \geq q_n\,|J_n|$.

PROPOSITION 6: *For $k > 1$ the distribution function with respect to Lebesgue measure of the k-th entrance time is given by*

$$(3) \qquad \Phi_n^{(k)}(t) = \sum_{i=0}^{q_{n-1}-1} \lambda\big(f^i(I_n \cap T^{-k} I_n)\big) + \sum_{j=0}^{q_n - 1} \lambda\big(f^j(I_{n-1} \cap T^{-k} I_n)\big)$$

if $q_{n-1} |J_n| \leq t < q_n |J_n|$; $\Phi_n^{(k)}(t) = 0$ *if* $t < q_{n-1} |J_n|$; *and* $\Phi_n^{(k)}(t) = 1$ *if* $t \geq q_n |J_n|$.

Proof: We use the same notation introduced in the proof of Proposition 4. Here we also need to compute the Lebesgue measure of the set E of the points $x \in S^1$ such that $D_n^{(k)}(x) = q_{n-1}$. From the arguments used in the proof of Proposition 4 and the fact that $T \equiv f^{q_{n-1}}$ on I_n, and $T \equiv f^{q_n}$ on I_{n-1}, we conclude that

$$E \cap f^i(I_n) = f^{-(q_{n-1}-i)}\big(T^{-(k-1)} I_n \cap T(I_n)\big) = f^i(T^{-k} I_n \cap I_n)$$

for $0 \leq i < q_{n-1}$, and

$$E \cap f^j(I_{n-1}) = f^{-(q_n - j)}\big(T^{-(k-1)} I_n \cap T(I_{n-1})\big) = f^j(T^{-k} I_n \cap I_{n-1})$$

for $0 \leq j < q_n$. Now the proof of this Proposition follows by noting that, except for a finite number of points, every point in S^1 belongs to exactly one of the intervals of the collection $\{f^i(I_n), f^j(I_{n-1})\}$ for $0 \leq i < q_{n-1}$ and $0 \leq j < q_n$. ∎

3. Proofs of Theorems I and II

Let $L_n(t)$ be the continuous piecewise linear approximation of $F_n^{(1)}(t)$ defined by $L_n(t) = 0$ if $t < 0$; $L_n(t) = t$ if $0 \leq t < q_{n-1}\,\mu(J_n)$;

$$(4) \qquad \begin{aligned} L_n(t) &= \frac{\mu(I_{n-1})}{\mu(J_n)} (t - q_{n-1}\,\mu(J_n)) + q_{n-1}\,\mu(J_n) \\ &= \frac{\mu(I_{n-1})}{\mu(J_n)} t + q_{n-1}\,\mu(I_n) \end{aligned}$$

if $q_{n-1}\,\mu(J_n) \leq t < q_n\,\mu(J_n)$; and $L_n(t) = 1$ if $t \geq q_n\,\mu(J_n)$.

LEMMA 7: *The sequence $F_n^{(1)}(t)$ converges pointwise (uniformly) if and only if $L_n(t)$ converges pointwise (uniformly).*

Proof: This is clear from the fact that $0 \leq L_n(t) - F_n^{(1)}(t) \leq \mu(J_n)$ for all t. ∎

LEMMA 8: *Let* $\{n_i\}$ *be a subsequence of* $\mathbb{N}$. *Then* $q_{n_i}\,\mu(J_{n_i})$ *converges to* 1 *if and only if* $\lim a_{n_i} = \infty$.

Proof: Let δ_n denote $|q_n\,\alpha - p_n|$. A basic result of continued fraction theory asserts that

$$\frac{1}{q_n + q_{n+1}} < \delta_n < \frac{1}{q_{n+1}}\,. \tag{5}$$

Since the semi-conjugacy between f and the rotation by α carries μ to the Lebesgue measure, we know that

$$\mu(J_n) = \mu(I_n) + \mu(I_{n-1}) = \delta_n + \delta_{n-1}\,,$$

which combined with Lemma 1 yields

$$q_n\,\mu(J_n) = 1 + (q_n - q_{n-1})\,\delta_n\,. \tag{6}$$

Now from (5) we note that

$$0 < (q_n - q_{n-1})\,\delta_n < \frac{q_n}{q_{n+1}} < \frac{1}{a_n}\,. \tag{7}$$

In particular, if $\lim a_{n_i} = \infty$ then $q_{n_i}\,\mu(J_{n_i})$ converges to 1.

In order to prove the converse we argue by contradiction. Let m_k be a subsequence of n_i such that $\lim a_{m_k} = a < \infty$. By (5) we always have

$$(q_n - q_{n-1})\,\delta_n > \frac{1 - \frac{q_{n-1}}{q_n}}{1 + \frac{q_{n+1}}{q_n}} > \frac{1 - \frac{1}{a_{n-1}}}{2 + a_n}\,. \tag{8}$$

Therefore along the sequence m_k the first member of (8) cannot go to zero unless $a_{m_k-1} = 1$ for all sufficiently large k. The same argument repeated inductively shows that $a_{m_k-\ell} = 1$ for all sufficiently large k (depending on ℓ). In particular $a_{m_k-1} = a_{m_k-2} = a_{m_k-3} = 1$ for all sufficiently large k. Since $q_n/q_{n+1} = [a_n, a_{n-1}, \ldots, a_0]$ we have

$$\frac{q_{m_k}}{q_{m_k-1}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \beta_k}}\,,$$

where $0 < \beta_k < 1$, and also

$$\frac{q_{m_k-1}}{q_{m_k-2}} = 1 + \frac{1}{1 + \beta_k}\,.$$

These two ratios lie between $\frac{3}{2}$ and 2. Therefore we obtain

$$1 - \frac{q_{m_k-1}}{q_{m_k}} > \frac{1}{9}\left(1 + \frac{q_{m_k+1}}{q_{m_k}}\right),$$

which is incompatible with (8). ∎

Proof of Theorem I: By Lemma 7, pointwise convergence of $F^{(1)}_{n_i}(t)$ is equivalent to pointwise convergence of $L_{n_i}(t)$. If $L_{n_i}(t)$ converges pointwise then in particular $\lim L_{n_i}(1) = b \leq 1$. For each $n > 0$ the interval $\Delta_n = [q_{n-1}\,\mu(J_n), q_n\,\mu(J_n)]$ contains 1 and $L_n(t)$ is affine on Δ_n with slope given by the ratio

$$\frac{\mu(I_{n-1})}{\mu(J_n)} = \frac{\delta_{n-1}}{\delta_n + \delta_{n-1}}, \tag{9}$$

which lies between $\frac{1}{2}$ and 1. Therefore if $b = 1$ then $q_{n_i}\,\mu(J_{n_i})$ converges to 1, and by Lemma 8 we have $\lim a_{n_i} = \infty$, i.e. $G^{n_i}(\alpha) \to 0$. However, if $b < 1$ then $\Delta_{n_i} \supseteq [1, 1 + \frac{(1-b)}{2}]$ for all sufficiently large i. This shows that the ratio

$$\frac{\delta_n}{\delta_{n-1}} = \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{\cdots}}} = G^n(\alpha), \tag{10}$$

converges along the subsequence σ to some $\theta > 0$, and it also shows that the intervals Δ_{n_i} converge to an interval containing $[1, 1 + \frac{(1-b)}{2}]$. Therefore the ratio $q_{n_i-1}/q_{n_i} = [a_{n_i-1}, a_{n_i-2}, \ldots, a_0]$ of their endpoints converges to some $\omega < 1$, and this happens if and only if the second coordinate of $\Gamma^{n_i}(\alpha, \cdot)$ converges to ω.

In order to prove the converse, we let us first deal with the case $G^{n_i}(\alpha) \to 0$. Let $U(t)$ denote the uniform distribution on $[0,1]$. Then we see that

$$0 \leq U(t) - L_n(t) \leq 1 - L_n(1), \tag{11}$$

for all t and $n > 0$. From Lemma 8 we know that $q_{n_i}\,\mu(J_{n_i})$ converges to 1, and so from the expression of $L_n(1)$ in (4) together with (9) and (10) we deduce that $\lim L_{n_i}(1) = 1$. Therefore by (11), $L_{n_i}(t)$ converges uniformly to $U(t)$, which sets the converse in this case and also proves (a).

We now deal with the case $\Gamma^{n_i}(\alpha, \cdot) \to (\theta, \omega)$ for some $\theta > 0$ and $\omega < 1$, or equivalently

$$\lim \frac{q_{n_i-1}}{q_{n_i}} = \omega, \quad \lim \frac{\delta_{n_i}}{\delta_{n_i-1}} = \theta. \tag{12}$$

It is clear from (9) that the slope of $L_{n_i}(t)$ on Δ_{n_i} converges to $(1+\theta)^{-1}$. Therefore in order to show that the sequence $L_{n_i}(t)$ converges uniformly it suffices to prove that the right hand endpoint of Δ_{n_i} has a limit. By (6) this is equivalent to showing that s_{n_i} has a limit, where $s_n = (q_n - q_{n-1})\,\delta_n$. However, by (7) s_{n_i} is a bounded sequence, and from the first inequality in (8) it is also bounded away from zero for sufficiently large i. Therefore s_{n_i} converges if and only if the ratio

$$\frac{1+s_{n_i}}{s_{n_i}} = \frac{q_{n_i}}{(q_{n_i} - q_{n_i-1})} \cdot \frac{(\delta_{n_i} + \delta_{n_i-1})}{\delta_{n_i}}$$

converges, and this is clear from (12). Now a simple computation shows that

$$\lim s_{n_i} = \frac{(1-\omega)\,\theta}{1+\theta\,\omega},$$

which implies

$$\lim q_{n_i-1}\,\mu(J_{n_i}) = \frac{(1+\theta)\,\omega}{1+\theta\,\omega} \tag{13}$$

and

$$\lim q_{n_i}\,\mu(J_{n_i}) = \frac{1+\theta}{1+\theta\,\omega}. \tag{14}$$

This completes the proof of the converse and also proves (b). ∎

Proof of Theorem II: Let $q_{n-1}\,\mu(J_n) \le t < q_n\,\mu(J_n)$. Then by Proposition 4 we have

$$F_n^{(k)}(t) = q_n\,\mu(J_n)\left[\frac{q_{n-1}}{q_n} \cdot \frac{\mu(I_n \cap T^{-k} I_n)}{\mu(J_n)} + \frac{\mu(I_{n-1} \cap T^{-k} I_n)}{\mu(J_n)}\right]. \tag{15}$$

In the case $G^{n_i}(\alpha) \to 0$ we know from the proof of Theorem I that $\lim q_{n_i}\,\mu(J_{n_i}) = 1$. Since the expression between brackets in (15) is always bounded by $\mu(I_n)/\mu(J_n)$ and this ratio converges to zero along the subsequence σ in this case, we have proved (a).

Now we deal with the case $\Gamma^{n_i}(\alpha,\cdot) \to (\theta,\omega)$ for some $\theta > 0$ and $\omega < 1$. Let h be the semi-conjugacy between f and the corresponding rotation R_α. Since h is an isomorphism between the ergodic systems (f,μ) and (R_α,λ), where λ denotes Lebesgue measure, we have

$$\frac{\mu(I_n \cap T^{-k} I_n)}{\mu(J_n)} = \frac{\lambda(h\,I_n \cap R_\alpha^{-k}(h\,I_n))}{\lambda(h\,J_n)},$$

and a similar equality holds for the other ratio in (15) involving I_{n-1}. Therefore, in order to prove (b), we may assume that f is the rotation by α and μ is Lebesgue measure.

Let I be the unit interval and let A: $J_n \to I$ be the unique affine orientation preserving map carrying J_n onto I. Consider the conjugate map ATA^{-1} defined on I. Through the identification of the points 0 and 1 via the canonical projection $\exp : \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$, this conjugate map becomes a new rotation $T_\star$ on the circle (the n^{th} renormalization of R_α). It can be shown that the rotation number of $T_\star$ is $\rho = [a_n+1, a_{n+1}, a_{n+2}, \ldots]$ (cf. [La, dF]). Writing $\Delta = \exp \circ A(I_n)$ we see that $\lambda(\Delta) = \rho$ and

$$\frac{\mu(I_n \cap T^{-k} I_n)}{\mu(J_n)} = \lambda(\Delta \cap T_\star^{-k} \Delta) . \tag{16}$$

Let w denote an endpoint of Δ and let d denote the intrinsic distance on the circle. Then

$$d(w, T_\star^{-k}(w)) = \|-k\,\rho\| = \|k\,\rho\| .$$

If $k > 1$ is such that $\|k\,\rho\| > \rho$ then Δ and $T_\star^{-k}(\Delta)$ are disjoint. Alternatively, if $\|k\,\rho\| < \rho$ then $\lambda(\Delta \cap T_\star^{-k}\Delta)$ is equal to

$$\lambda(\Delta) - d(w, T_\star^{-k}(w)) = \rho - \|k\,\rho\| .$$

In both cases, we obtain

$$\lambda(\Delta \cap T_\star^{-k}\Delta) = \rho - \min\{\rho, \|k\,\rho\|\} . \tag{17}$$

We now note that along the subsequence σ the corresponding rotation numbers ρ converge to $\gamma = \theta/(1+\theta)$. Therefore, letting $n \to \infty$ along σ in (15) and taking into account the expressions (13), (14), (16) and (17) we deduce that

$$\lim F_{n_i}^{(k)}(t) = \frac{1+\theta}{1+\theta\,\omega}\left[\omega\left(\gamma - \min\{\gamma, \|k\,\gamma\|\}\right) + \gamma - \left(\gamma - \min\{\gamma, \|k\,\gamma\|\}\right)\right] ,$$

for all $\frac{(1+\theta)\,\omega}{1+\theta\,\omega} \le t < \frac{1+\theta}{1+\theta\,\omega}$. This completes the proof of (b). ∎

4. Proof of Theorem III

In the proof of Theorem III, we will consider the random variables

$$\hat{D}_n^{(k)}(x) = \mu(J_n)\left(N_n^{(k)}(x) - N_n^{(k-1)}(x)\right),$$

and their joint distributions

$$F_n^{(k_1,\dots,k_s)}(t_1,\dots,t_s) = \mu\{x \in S^1\colon\ \hat{D}_n^{(k_i)}(x) \le t_i\,,\ i=1,\dots,s\}\,, \tag{18}$$

where $1 \le k_1 < \cdots < k_s$. We will also need the following Lemma, which can be proved by a renormalization argument similar to the one used to obtain equality (16).

LEMMA 9: *Let $\sigma = \{n_i\}$ be a subsequence of $\mathbb{N}$ such that $G^{n_i}(\alpha) \to \theta$ with $\theta \ge 0$. Then for every $1 \le k_1 < \cdots < k_s$ we have*

$$\lim_{\sigma \ni n \to \infty} \frac{\mu(I_n \cap T^{-k_1} I_n \cap \cdots \cap T^{-k_s} I_n)}{\mu(J_n)} = \lambda(\Delta_\theta \cap {T_\theta}^{-k_1}\Delta_\theta \cap \cdots \cap {T_\theta}^{-k_s}\Delta_\theta)\,,$$

where $\Delta_\theta = [0, \theta/(1+\theta)] \subseteq \mathbb{R}/\mathbb{Z}$ and T_θ is the rotation by $\theta/(1+\theta)$.

Proof of Theorem III: We note that convergence in law of the point process τ_n is equivalent to convergence in law of the joint distributions given by (18), for every choice of indices $1 \le k_1 < \cdots < k_s$ (cf. [Ne], page 284). Since in our case the corresponding probability measures in $\mathbb{R}^s$ defined by these joint distributions are supported in the cube $[0, q_n\,\mu(J_n)]^s \subseteq [0,2]^s$ for all n, convergence in law of these distributions is equivalent to their pointwise convergence. In particular, if τ_n converges in law along the subsequence σ then the individual distributions of $\hat{D}_n^{(k)}$ converge pointwise. Therefore, by Theorems I and II, either $G^{n_i}(\alpha) \to 0$ or $\Gamma^{n_i}(\alpha,\cdot) \to (\theta,\omega)$ for some $\theta > 0$ and $\omega < 1$.

Now we prove the converse. Recall that $\hat{D}_n^{(k)}$ for $k \ge 2$ assumes only the values $q_{n-1}\,\mu(J_n)$ and $q_n\,\mu(J_n)$. Therefore in order to prove convergence of the joint distributions in (18) along σ, it suffices to determine whether the limit of

$$F_n^{(k_1,\dots,k_s)}\left(t, q_{n-1}\,\mu(J_n), \dots, q_{n-1}\,\mu(J_n)\right) \tag{19}$$

exists along σ, for every $t > 0$. In fact, if $k_1 > 1$ it suffices to prove the existence of the limit of (19) along σ for $t = q_{n-1}\,\mu(J_n)$.

Let us first deal with the case $k_1 = 1$. Consider the following conditions

$$\hat{D}_n^{(1)}(x) \le t\,,\ \hat{D}_n^{(k_2)}(x) = q_{n-1}\,\mu(J_n)\,,\ \dots,\ \hat{D}_n^{(k_s)}(x) = q_{n-1}\,\mu(J_n)\,, \tag{20}$$

for a fixed $t > 0$ and let $r\,\mu(J_n) \le t < (r+1)\,\mu(J_n)$ for some integer $r \ge 0$. There are three cases to consider: $0 \le r < q_{n-1}$, $q_{n-1} \le r < q_n$ and $r \ge q_n$. Using Lemma 1 and Proposition 2 we deduce the following. In the first case, a point x satisfies conditions (20) if and only if it belongs to

$$\bigcup_{i=0}^{r} f^{q_{n-1}-i}(T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n) .$$

In the second case, x satisfies (20) if and only if it belongs to

$$\bigcup_{i=0}^{q_{n-1}-1} f^{q_{n-1}-i}(T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n) \cup \bigcup_{j=q_{n-1}}^{r} f^{q_n-j}(I_{n-1} \cap T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n) .$$

Finally, in the third case, every x satisfies the first inequality in (20) and we are reduced to the case $k_1 > 1$, which will be dealt with below.

Therefore, in the case $0 \le r < q_{n-1}$ we obtain

$$F_n^{(1,k_2,\ldots,k_s)}\big(t, q_{n-1}\,\mu(J_n), \ldots, q_{n-1}\,\mu(J_n)\big) = (r+1)\,\mu(J_n)\, \frac{\mu(T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n)}{\mu(J_n)} .$$

If we let $n \to \infty$ along a subsequence σ satisfying either (a) or (b) in the statement of Theorem III, we deduce from Lemma 9 that the last expression converges to

$$t\,\lambda({T_\theta}^{-k_2} \Delta_\theta \cap \cdots \cap {T_\theta}^{-k_s} \Delta_\theta) .$$

In the case $q_{n-1} \le r < q_n$ we have

$$\begin{aligned} &F_n^{(1,k_2,\ldots,k_s)}\big(t, q_{n-1}\,\mu(J_n), \ldots, q_{n-1}\,\mu(J_n)\big) \\ &\quad = q_{n-1}\,\mu(T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n) \\ &\qquad + (r - q_{n-1} + 1)\,\mu(I_{n-1} \cap T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n) \\ &\quad = (r+1)\,\mu(J_n)\, \frac{\mu(T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n)}{\mu(J_n)} \\ &\qquad - (r - q_{n-1} + 1)\,\mu(J_n)\, \frac{\mu(I_n \cap T^{-k_2} I_n \cap \cdots \cap T^{-k_s} I_n)}{\mu(J_n)} . \end{aligned}$$

Again by Lemma 9, as $n \to \infty$ along σ, the above expression converges to

$$t\,\lambda({T_\theta}^{-k_2} \Delta_\theta \cap \cdots \cap {T_\theta}^{-k_s} \Delta_\theta) - \Big(t - \frac{(1+\theta)\,\omega}{1+\theta\,\omega}\Big)\,\lambda(\Delta_\theta \cap {T_\theta}^{-k_2} \Delta_\theta \cap \cdots \cap {T_\theta}^{-k_s} \Delta_\theta) .$$

Now suppose $k_1 > 1$. Here we need to determine whether the limit of (19) exists along σ for $t = q_{n-1}\,\mu(J_n)$. Applying Lemma 1 and Proposition 2 once more, we deduce that a point x satisfies

$$\hat{D}_n^{(k_1)}(x) = q_{n-1}\,\mu(J_n)\,,\ \ldots,\ \hat{D}_n^{(k_s)}(x) = q_{n-1}\,\mu(J_n)\,,$$

if and only if it belongs to

$$\bigcup_{i=0}^{q_{n-1}-1} f^{q_{n-1}-i}(T^{-k_1}I_n\cap\cdots\cap T^{-k_s}I_n)\cup\bigcup_{j=q_{n-1}}^{q_n-1} f^{q_n-j}(I_{n-1}\cap T^{-k_1}I_n\cap\cdots\cap T^{-k_s}I_n)\ .$$

Therefore we obtain

$$\begin{aligned} F_n^{(k_1,\ldots,k_s)}&\big(q_{n-1}\,\mu(J_n),\ldots,q_{n-1}\,\mu(J_n)\big)\\ &= q_{n-1}\,\mu(T^{-k_1}I_n\cap\cdots\cap T^{-k_s}I_n)\\ &\qquad + (q_n - q_{n-1})\,\mu(I_{n-1}\cap T^{-k_1}I_n\cap\cdots\cap T^{-k_s}I_n)\ . \end{aligned}$$

As before, using Lemma 9 we conclude that the limit of the above expression as $n\to\infty$ along σ equals

$$\frac{1+\theta}{1+\theta\omega}\left[\lambda({T_\theta}^{-k_1}\Delta_\theta\cap\cdots\cap{T_\theta}^{-k_s}\Delta_\theta) - (1-\omega)\,\lambda(\Delta_\theta\cap{T_\theta}^{-k_1}\Delta_\theta\cap\cdots\cap{T_\theta}^{-k_s}\Delta_\theta)\right]\ .$$

Therefore, we have proved that all joint distributions converge along a subsequence σ, which satisfies either (a) or (b) in the statement of Theorem III. In fact, with the given expressions, it is possible to write explicit formulae for all limit joint distributions. We note that when $\theta = 0$ the limit joint distributions are a product of the individual limit distributions. Therefore the limiting process is independent in this case. ∎

5. Proofs of Theorems IV and V

Throughout this section, let f be a diffeomorphism which is C^1-conjugate to a rotation with irrational rotation number α. Let $\Lambda_n(t)$ be the continuous piecewise linear function defined by $\Lambda_n(t) = 0$ if $t < 0$; $\Lambda_n(t) = (\mu(J_n)/|J_n|)\,t$ if $0 \le t < q_{n-1}\,|J_n|$;

$$\begin{aligned} \Lambda_n(t) &= \frac{\mu(I_{n-1})}{|J_n|}\,(t - q_{n-1}\,|J_n|) + q_{n-1}\,\mu(J_n)\\ &= \frac{\mu(I_{n-1})}{|J_n|}\,t + q_{n-1}\,\mu(I_n) \end{aligned} \tag{21}$$

if $q_{n-1}\,|J_n| \le t < q_n\,|J_n|$; and $\Lambda_n(t) = 1$ if $t \ge q_n\,|J_n|$.

LEMMA 10: *The sequence $\Phi_n^{(1)}(t)$ converges pointwise (uniformly) if and only if $\Lambda_n(t)$ converges pointwise (uniformly).*

Proof: Let $\Theta_n(t)$ denote the difference $\left|\Phi_n^{(1)}(t)-\Lambda_n(t)\right|$. If t is such that $k\,|J_n| \le t < (k+1)|J_n|$ for some $0 \le k < q_{n-1}$, then

$$\begin{aligned}\Theta_n(t) &= \left|\sum_{i=0}^{k} |f^i(J_n)| - \frac{\mu(J_n)}{|J_n|}\,t\right| \\ &\le \left|\frac{1}{\mu(J_n)}\int_{J_n}\frac{1}{(k+1)}\sum_{i=0}^{k} Df^i(x)\,dx - \frac{t}{(k+1)|J_n|}\right| \\ &= \left|\frac{1}{\mu(J_n)}\int_{J_n}\big(g(x)+\xi_n(t,x)\big)\,dx - \frac{t}{(k+1)|J_n|}\right| \\ &\le \|\xi_n(t,\cdot)\|_{C^0} + \left|1-\frac{t}{(k+1)|J_n|}\right|,\end{aligned}$$

where $\xi_n(t,x) = \big(1/(k+1)\big)\sum_{i=0}^{k} Df^i(x) - g(x)$. Since $(1/n)\sum_{i=0}^{n-1} Df^i(x)$ converges uniformly to $g(x)$ (cf. [He], Proposition IV.5.1.2) we conclude that $\|\xi_n(t,\cdot)\|$ converges uniformly to zero for $0 < t < q_{n-1}\,|J_n|$ and hence $\Theta_n(t)$ also converges uniformly to zero on this interval. However, if t satisfies $k\,|J_n| \le t < (k+1)|J_n|$ for some $q_{n-1} \le k < q_n$, then

$$\begin{aligned}\Theta_n(t) &\le \left|\sum_{j=0}^{k}|f^j(I_{n-1})| - \frac{\mu(I_{n-1})}{|J_n|}\,t\right| + \left|\sum_{i=0}^{q_{n-1}-1}|f^i(I_n)| - q_{n-1}\,\mu(I_n)\right| \\ &\le \left|\frac{1}{\mu(I_{n-1})}\int_{I_{n-1}}\frac{1}{(k+1)}\sum_{j=0}^{k} Df^j(x)\,dx - \frac{t}{(k+1)|J_n|}\right| \\ &\quad + \left|\frac{1}{\mu(I_n)}\int_{I_n}\frac{1}{q_{n-1}}\sum_{i=0}^{q_{n-1}-1} Df^i(x)\,dx - 1\right|,\end{aligned}$$

and by a similar argument to the one used above we conclude that $\Theta_n(t)$ also converges uniformly to zero in this case. ∎

Proof of Theorem IV: Let $L_n(t)$ be the function introduced in (4). We note that $\Lambda_n(t) = L_n\big(t\,\mu(J_n)/|J_n|\big)$. Since the density $g(x)$ of μ is a continuous function, we know that the ratio $\mu(J_n)/|J_n|$ converges to $g(z)$. Consequently, if $L_{n_i}(t)$ converges to some $F_\sigma(t)$, then $\Lambda_{n_i}(t)$ converges to $F_\sigma\big(g(z)\,t\big)$. Therefore Theorem IV follows from Lemma 10 and Theorem I. ∎

Proof of Theorem V: Let $k > 1$ be fixed and consider the difference

$$\Psi_n(t) = \left|\Phi_n^{(k)}(t) - F_n^{(k)}\Big(\frac{\mu(J_n)}{|J_n|}\,t\Big)\right| .$$

This Theorem will follow from Theorem II if we show that $\Psi_n(t)$ converges uniformly to zero. It suffices to consider $q_{n-1}\,|J_n| \le t < q_n\,|J_n|$, since Ψ_n vanishes outside this interval. Comparing the last two summands in (2) and (3) we have

$$\Big|\sum_{j=0}^{q_n-1}\lambda\big(f^j(I_{n-1}\cap T^{-k}I_n)\big) - q_n\,\mu(I_{n-1}\cap T^{-k}I_n)\Big|$$

$$= q_n\,\mu(J_n)\left|\frac{1}{\mu(J_n)}\int_{I_{n-1}\cap T^{-k}I_n}\frac{1}{q_n}\sum_{j=0}^{q_n-1} Df^j(x)\,dx - \frac{\mu(I_{n-1}\cap T^{-k}I_n)}{\mu(J_n)}\right|$$

$$= q_n\,\mu(J_n)\left|\frac{1}{\mu(J_n)}\int_{I_{n-1}\cap T^{-k}I_n}\big(g(x)+\eta_n(x)\big)\,dx - \frac{\mu(I_{n-1}\cap T^{-k}I_n)}{\mu(J_n)}\right|$$

$$\le q_n\,\|\eta_n\|_{C^0}\,\lambda(I_{n-1}\cap T^{-k}I_n) \;\le\; q_n\,|I_{n-1}|\,\|\eta_n\|_{C^0}\ ,$$

where $\eta_n(x) = (1/q_n)\sum_{j=0}^{q_n-1} Df^j(x) - g(x)$. Since $q_n\,|I_{n-1}|$ is a bounded sequence and $\|\eta_n\|_{C^0}$ converges to zero, the above difference also converges to zero. The same argument implies that the difference between the first summands of (2) and (3) converges to zero. This completes the proof of the Theorem. ∎

ACKNOWLEDGEMENT: The first named author thanks Pierre Collet for invaluable discussions on entrance time processes and for originally pointing out this problem. Both authors are grateful to Pablo Ferrari, Antonio Galves and Servet Martinez for useful discussions on renewal processes, and to Benjamin Weiss for important comments concerning Theorem III.

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